ON A GENERAL ANALYTICAL FORMULA FOR $U_q(su(3))$-CLEBSCH-GORDAN COEFFICIENTS

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Abstract

We present the projection operator method in combination with the Wigner-Racah calculus of the subalgebra $U_q(su(2))$ for calculation of Clebsch-Gordan coefficients (CGCs) of the quantum algebra $U_q(su(3))$. The key formulas of the method are couplings of the tensor and projection operators and also a tensor form for the projection operator of $U_q(su(3))$. We obtain a very compact general analytical formula for the $U_q(su(3))$ CGCs in terms of the $U_q(su(2))$ Wigner $3nj$-symbols.

1 Introduction

It is well known that the Clebsch-Gordan coefficients (CGCs) of the unitary Lie algebra $u(n)$ ($su(n)$) have numerous applications in various fields of theoretical and mathematical physics. For example, many algebraic models of nuclear theory(interacting boson model (IBM), Elliott $su(3)$ model, $su(4)$ supermultiplet scheme of Wigner, the shell model, and so on) demand the CGCs for $su(6)$, $su(5)$, $su(3)$, $su(4)$ and $su(n)$. Analogously, in quark models of hadrons we need the CGCs of $su(3)$, $su(4)$, etc. The theory of the $su(n)$ CGCs is connected with the theory of special functions, combinatorial analysis, topology, etc.

There are several methods for the calculation of CGCs of $su(n)$ ($u(n)$) and other Lie algebras: recursion method; method of employment of explicit bases of irreducible representations; method of generating invariants; method of tensor operators, where the Wigner-Eckart theorem is used; projection operator method; coherent state method; combined methods.

It is well known that the method of projection operators for usual (non-quantized) Lie algebras [1, 2] and superalgebras [2] is powerful and universal method for a solution of many

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problems in the representation theory. In particular, the method allows to develop the detailed theory of Clebsch-Gordan coefficients and another elements of Wigner-Racah calculus (including compact analytic formulas of these elements and their symmetry properties) [3] and so on. It is evident that the projection operators of quantum groups [4] play the same role in their representation theory.

In this paper we present the projection operator method in combination with the Wigner-Racah calculus of the subalgebra $U_q(su(2))$ [5] for calculation of CGCs of the quantum algebra $U_q(su(3))$. The key formulas of the method are couplings of the tensor and projection operators and also a tensor form for the projection operator of $U_q(su(3))$. It should be noted that the first application of this method was for the $su(3)$ case in [3]. Some simple elements of this approach were also used in [6] for the $U_q(su(n))$ case. Also, the coherent state method in combination with the Wigner-Racah calculus was applied in [7] for $u(n)$.

2 Gelfand-Tsetlin basis

Let $\Pi := \{\alpha_1, \alpha_2\}$ be a system of simple roots of the Lie algebra $sl(3)$ $(= sl(3, \mathbf{C}) \simeq A_2)$, endowed with the following scalar product: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -1$. The root system Δ_+ of $sl(3)$ consists of the roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$. The quantum Hopf algebra $U_q(sl(3))$ is generated by the Chevalley elements $q^{\pm h_{\alpha_i}}$, $e_{\pm\alpha_i}$ $(i = 1, 2)$ with the relations:

$$\begin{aligned} q^{h_{\alpha_i}} q^{-h_{\alpha_i}} &= q^{-h_{\alpha_i}} q^{h_{\alpha_j}} = 1, \qquad q^{h_{\alpha_i}} q^{h_{\alpha_j}} = q^{h_{\alpha_j}} q^{h_{\alpha_i}}, \qquad q^{h_{\alpha_i}} e_{\alpha_j} q^{-h_{\alpha_i}} = q^{(\alpha_i, \alpha_j)} e_{\alpha_j}, \\ [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{ij} \, [h_{\alpha_i}] \;, \qquad [[e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q, e_{\pm\alpha_j}]_q = 0 \qquad \text{for } \; |i - j| = 1 \;. \end{aligned} \tag{2.1}$$

Here and elsewhere we use the standard notation $[a] := (q^a - q^{-a})/(q - q^{-1})$, and $[e_\alpha, e_\beta]_q := e_\alpha e_\beta - q^{(\alpha,\beta)} e_\beta e_\alpha$. The Hopf structure of $U_q(u(3))$ is given by

$$\begin{aligned} \Delta_q(h_{\alpha_i}) &= h_{\alpha_i} \otimes 1 + 1 \otimes h_{\alpha_i}, & S_q(h_{\alpha_i}) &= -h_{\alpha_i}, \\ \Delta_q(e_{\pm\alpha_i}) &= e_{\pm\alpha_i} \otimes q^{\frac{1}{2} h_{\alpha_i}} + q^{-\frac{1}{2} h_{\alpha_i}} \otimes e_{\pm\alpha_i}, & S_q(e_{\pm\alpha_i}) &= -q^{\pm 1} e_{\pm\alpha_i}. \end{aligned} \tag{2.2}$$

For construction of the composite root vectors $e_{\pm(\alpha_1+\alpha_2)}$ we fix the normal ordering in Δ_+: $\alpha_1, \ \alpha_1 + \alpha_2, \ \alpha_2$. According to this ordering we put

$$e_{\alpha_1+\alpha_2} := [e_{\alpha_1}, e_{\alpha_2}]_{q^{-1}} \;, \qquad e_{-\alpha_1-\alpha_2} := [e_{-\alpha_2}, e_{-\alpha_1}]_q \;. \tag{2.3}$$

Let us introduce another standard notations for the Cartan-Weyl generators:

$$\begin{aligned} e_{12} &:= e_{\alpha_1}, & e_{21} &:= e_{-\alpha_1}, & e_{11} - e_{22} &:= h_{\alpha_1}, \\ e_{23} &:= e_{\alpha_2}, & e_{32} &:= e_{-\alpha_2}, & e_{22} - e_{33} &:= h_{\alpha_2}, \\ e_{13} &:= e_{\alpha_1+\alpha_2}, & e_{31} &:= e_{-\alpha_1-\alpha_2}, & e_{11} - e_{33} &:= h_{\alpha_1} + h_{\alpha_2}. \end{aligned} \tag{2.4}$$

The explicit formula for the extremal projector for the quantum groups [4] specialized to the case of $U_q(sl(3))$ has the form

$$p(U_q(sl(3))) = p_{12} p_{13} p_{23} \;, \tag{2.5}$$

where the elements p_{ij} $(1 \leq i < j \leq 3)$ are given by

$$p_{ij} = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]!} \varphi_{ij,n} \, e_{ij}^n e_{ji}^n \ , \qquad \varphi_{ij,n} = q^{-(j-i-1)n} \Big\{ \prod_{s=1}^{n} [e_{ii} - e_{jj} + j - i + s] \Big\}^{-1} . \tag{2.6}$$

The extremal projector $p := p(U_q(sl(3)))$ satisfies the relations:

$$e_{ij} p = p e_{ji} = 0 \quad (i < j) \ , \qquad p^2 = p \ . \tag{2.7}$$

The quantum algebra $U_q(su(3))$ can be considered as the quantum algebra $U_q(sl(3))$ endowed with the additional Cartan involution $(^*)$:

$$h_{\alpha_i}^* = h_{\alpha_i} \ , \qquad e_{\pm\alpha_i}^* = e_{\mp\alpha_i} \ , \qquad q^* = q \ (\text{or } \bar{q} := q^{-1}) \ . \tag{2.8}$$

Let $(\lambda\mu)$ be a finite-dimensional irreducible representation (IR) of $U_q(su(3))$ with the highest weight $(\lambda\mu)$ (λ and μ are nonnegative integers). The vector of the highest weight, denoted by the symbol $|(\lambda\mu)h\rangle$, satisfy the relations

$$h_{\alpha_1} |(\lambda\mu)h\rangle = \lambda |(\lambda\mu)h\rangle \ , \quad h_{\alpha_2} |(\lambda\mu)h\rangle = \mu |(\lambda\mu)h\rangle \ , \quad e_{ij} |(\lambda\mu)h\rangle = 0 \quad (i < j) \ . \tag{2.9}$$

Labelling of another basis vectors in IR $(\lambda\mu)$ depends upon choice of subalgebras of $U_q(su(3))$ (or in another words, depends upon which reduction chain from $U_q(su(3))$ to subalgebras is chosen). Here we use the Gelfand-Tsetlin reduction chain:

$$U_q(su(3)) \supset U_q(u_Y(1)) \otimes U_q(su_T(2)) \supset U_q(u_{T_0}(1)) \ , \tag{2.10}$$

where the subalgebra $U_q(su_T(2))$ is generated by the elements

$$T_+ := e_{23} \ , \qquad T_- := e_{32} \ , \qquad T_0 := \tfrac{1}{2}(e_{22} - e_{33}) \ , \tag{2.11}$$

the subalgebra $U_q(u_{T_0}(1))$ is generated by q^{T_0}, and $U_q(u_Y(1))$ is generated by q^Y (In the classical (non-deformed) case in the elementary particle theory the subalgebra $su_T(2)$ is called the T-spin algebra and the element Y is the hypercharge operator), where:

$$Y = -\tfrac{1}{3}(2h_{\alpha_1} + h_{\alpha_2}) \ . \tag{2.12}$$

In the case of the reduction chain (2.10) the basis vectors of IR $(\lambda\mu)$ are denoted by

$$|(\lambda\mu)jtt_z\rangle \ . \tag{2.13}$$

Here the set jtt_z characterize the hypercharge Y and the T-spin and its projection:

$$\begin{aligned} q^{T_0} |(\lambda\mu)jtt_z\rangle \ &= \ q^{t_z} |(\lambda\mu)jtt_z\rangle \ , \qquad q^Y |(\lambda\mu)jtt_z\rangle = q^y |(\lambda\mu)jtt_z\rangle \ , \\ T_\pm |(\lambda\mu)jtt_z\rangle \ &= \ \sqrt{[t \mp t_z][t \pm t_z + 1]} |(\lambda\mu)jtt_z \pm 1\rangle \ , \end{aligned} \tag{2.14}$$

where the parameter j is connected with the eigenvalue y of the operator Y as follows $y = -\frac{1}{3}(2\lambda + \mu) + 2j$. It is not hard to show that the orthonormalized vectors (2.13) can be represented in the following form

$$|(\lambda\mu)jtt_z\rangle = N_{jt}^{(\lambda\mu)} P_{t_z;t}^{t}\, e_{31}^{j+\frac{1}{2}\mu-t} e_{21}^{j-\frac{1}{2}\mu+t} |(\lambda\mu)h\rangle \ , \tag{2.15}$$

where $P^t_{t_z;t'_z}$ is the general projection operator of the quantum algebra $U_q(su_T(2))$ [5], and the normalizing factor $N_{jt}^{(\lambda\mu)}$ has the form

$$N_{jt}^{(\lambda\mu)} = \left(\frac{[\lambda + \frac{1}{2}\mu - j + t + 1]![\lambda + \frac{1}{2}\mu - j - t]![\frac{1}{2}\mu + j + t + 1]![\frac{1}{2}\mu - j + t]!}{q^{2j+\mu-2t}\,[\lambda]![\mu]![\lambda + \mu + 1]![j + \frac{1}{2}\mu - t]![j - \frac{1}{2}\mu + t]![2t + 1]!}\right)^{\frac{1}{2}} . \tag{2.16}$$

The quantum numbers jt are taken all nonnegative integers and half-integers such that the sum $\frac{1}{2}\mu + j + t$ is an integer and they are subjected to the constraints:

$$\begin{cases} \frac{1}{2}\mu + j - t \ \geq \ 0 \ , & -\frac{1}{2}\mu + j + t \geq 0 \ , \\ \frac{1}{2}\mu - j + t \ \geq \ 0 \ , & \frac{1}{2}\mu + j + t \geq \lambda + \mu \ . \end{cases} \tag{2.17}$$

For every fixed t the projection t_z runs the values $t_z = -t, -t+1, \ldots, t-1, t$. These results can be obtained from the explicit form of the Gelfand-Tsetlin bases for the case $U_q(su(n))$ [4] specializing to the given case $U_q(su(3))$.

3 Couplings of tensor and projection operators

Let $\{R^{j(q)}_{j_z}\}$ be an irreducible tensor operator (ITO) of the rank j, that is $(2j+1)$-components $R^{j(q)}_{j_z}$ are transformed with respect to the $U_q(su_T(2))$ adjoint action as the $U_q(su_T(2))$ basis vectors $|jj_z\rangle$ of the spin j:

$$T_i \triangleright R^{j(q)}_{j_z} := (\mathrm{ad}_q\, T_i) R^{j(q)}_{j_z} \equiv ((\mathrm{id} \otimes S_q)\Delta_q(T_i)) \circ R^{j(q)}_{j_z} = \sum_{j'_z} \langle jj'_z|T_i|jj_z\rangle R^{j(q)}_{j'_z} \ , \tag{3.1}$$

where $(a \otimes b) \circ x = axb$. The tensor operator of the type $\{R^{j(q)}_{j_z}\}$ will be also called the left irreducible tensor operators (LITO) because the generators T_i $(i = \pm, 0)$ act to the left-side of the components $R^{j(q)}_{j_z}$. (The given denotation of the ITOs is differed from one of the papers [5] by the replacement q by q^{-1}). Following to the paper [3] we also introduce a right irreducible tensor operator (RITO) denoted by the tilde symbol $\{\tilde{R}^{j(q)}_{j_z}\}$, on which the $U_q(su_T(2))$ generators T_i act on the right-side, namely

$$T_i \triangleleft \tilde{R}^{j(q)}_{j_z} := (\mathrm{ad}^*_q\, T_i)\, \tilde{R}^{j(q)}_{j_z} \equiv \tilde{R}^{j(q)}_{j_z} \overset{\leftarrow}{\circ} ((\tilde{S}_q \otimes \mathrm{id})\tilde{\Delta}_q(T^*_i)) = \sum_{j'_z} \langle jj'_z|T_i|jj_z\rangle \tilde{R}^{j(q)}_{j'_z} \ . \tag{3.2}$$

where $x \overset{\leftarrow}{\circ} (a \otimes b) = axb$, and $\tilde{\Delta}_q$ is the opposite coproduct $(\tilde{\Delta}_q = \Delta_{\bar{q}})$ and $\tilde{S}_q$ is the corresponding antipode $(\tilde{S}_q = S_{\bar{q}})$. It is not hard to verify that any LITO $\{R^{j(q)}_{j_z}\}$ is the

RITO $\{\tilde{R}^{j(q)}_{j_z}\}$: $R^{j(q)}_{j_z}=(-1)^{j_z}q^{j_z}\tilde{R}^{j(q)}_{-j_z}$. The projection operator set $\{P^t_{t_z;t'_z}\}$ for a fixed IR t and for various t_z and t'_z will be called the $\mathbf{P}^t$-operator. It is not hard to see that the subset of the left components of this operator satisfy the relations for the LITO $\mathbf{R}^{j(q)}:=\{R^{j(q)}_{j_z}\}$ if we understand the action "$\triangleright$" of the generator T_i as the usual multiplication of the operators T_i and $P^t_{t_z;t'_z}$ and the subset of the right components of the $\mathbf{P}^t$-operator satisfy the relations for the RITO $\tilde{\mathbf{R}}^j:=\{\tilde{R}^{j(q)}_{j_z}\}$ if we understand the action "$\triangleleft$" as the usual multiplication of the operators $P^t_{t_z;t'_z}$ and T^*_i:

$$T_i\triangleright P^t_{t_z;t'_z}:=T_iP^t_{t_z:t'_z}=\sum_{t''_z}\langle tt''_z|T_i|tt_z\rangle P^t_{t''_z;t'_z}\ , \tag{3.3}$$

$$T_i\triangleleft P^t_{t_z;t'_z}:=P^t_{t_z:t'_z}T^*_i=\sum_{t''_z}\langle tt''_z|T_i|tt'_z\rangle P^t_{t_z;t''_z}\ . \tag{3.4}$$

Using the $U_q(su_T(2))$ CGCs we can couple the LITO $\mathbf{R}^{j(q)}$ with the left components of the $\mathbf{P}^t$-operator and the RITO $\tilde{\mathbf{R}}^{j(q)}$ with the right components of the $\mathbf{P}^t$-operator:

$${}^{t'}_{t'_z}\Big\{\mathbf{R}^{j(q)}\dot{\otimes}\mathbf{P}^t_{;t_z}\Big\}_q:=\sum_{j_zt''_z}(jj_z\,tt''_z|t't'_z)_q\,R^{j(q)}_{j_z}P^t_{t''_z,t_z}\ , \tag{3.5}$$

$$\Big\{\mathbf{P}^t_{t_z;}\dot{\otimes}\tilde{\mathbf{R}}^{j(q)}\Big\}^{t'}_{q\,t'_z}:=\sum_{j_zt''_z}(jj_z\,tt''_z|t't'_z)_q\,P^t_{t_z,t''_z}\tilde{R}^{j(q)}_{j_z}\ . \tag{3.6}$$

Here the symbol $\dot{\otimes}$ means that we first take the usual tensor product and then in a resulting expression we replace the tensor product by the usual operator product. It is not hard to show that the couplings (3.5) and (3.6) are connected as follows

$$\mathbb{R}^{j(q)}_{tt_z;t't'_z}:=\sqrt{[2t+1]}\ {}^{t}_{t_z}\Big\{\mathbf{R}^{j(q)}\dot{\otimes}\mathbf{P}^{t'}_{;t'_z}\Big\}_q=(-1)^{t'-t}\sqrt{[2t'+1]}\ \Big\{\mathbf{P}^t_{t_z;}\dot{\otimes}\tilde{\mathbf{R}}^{j(q)}\Big\}^{t'}_{q\,t'_z}\ . \tag{3.7}$$

Using (3.7) and an unitary relation of the $U_q(su(2))$ CGCs [5] one can obtain the following useful permutation relations between the components of the tensors $\mathbf{R}^{j(q)}$, $\tilde{\mathbf{R}}^{j(q)}$ and $\mathbf{P}^t$-operator:

$$R^{j(q)}_{j_z}P^t_{t_z;t'_z}=\sum_{t''t''_z}(-1)^{t-t''}\sqrt{\frac{[2t+1]}{[2t''+1]}}\,(jj_z\,tt_z|t''t''_z)_q\,\Big\{\mathbf{P}^{t''}_{t''_z;}\dot{\otimes}\tilde{\mathbf{R}}^{j(q)}\Big\}^{t}_{q\,t'_z}\ , \tag{3.8}$$

$$P^t_{t_z;t'_z}\tilde{R}^{j(q)}_{j_z}=\sum_{t''t''_z}(-1)^{t-t''}\sqrt{\frac{[2t+1]}{[2t''+1]}}\,(jj_z\,tt'_z|t''t''_z)_q\ {}^{t}_{t_z}\Big\{\mathbf{R}^{j(q)}\dot{\otimes}\mathbf{P}^{t''}_{;t''_z}\Big\}_q\ . \tag{3.9}$$

We can show that the monomials $e^n_{21}e^m_{31}$ and $e^n_{12}e'^m_{13}$ are components of ITOs with respect to the adjoint action of the subalgebra $U_q(su_T(2))$:

$$R^{j(q)}_{j_z}=\sqrt{\frac{[2j]!}{[j-j_z]![j+j_z]!}}\,q^{2j^2-j}\,e^{j+j_z}_{21}e^{j-j_z}_{31}q^{-jh_{\alpha_1}-(j-j_z)T_0}, \tag{3.10}$$

$$R'^{j(q)}_{j_z} = \sqrt{\frac{[2j]!}{[j-j_z]![j+j_z]!}}\; q^{-2j^2+j}\, e_{12}^{j-j_z} e'^{j+j_z}_{13} q^{-jh_{\alpha_1}-(j+j_z)T_0}. \tag{3.11}$$

where the generator e'_{13} is defined according to the inverse normal ordering: $\alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1$, i.e. $e'_{13}=[e_{23},e_{12}]_{q^{-1}}$. These ITOs have the remarkable properties: *A result of the coupling of two ITOs of the type (3.10) or (3.11) is non-zero only for an irreducible component of the maximal rank, e.g.*

$$\big\{\mathbf{R}^{j(q)}\dot{\otimes}\,\mathbf{R}^{j'(q)}\big\}_{q\,j''_z}^{j''} = \delta_{j'',j+j'}\, R^{j+j'(q)}_{j''_z}. \tag{3.12}$$

The property is also useful in applications: *For ITOs of the type (3.12) the relation is valid*

$$R^{j(q)}_{j_z}\mathbf{IR}^{j'(q)}_{tt_z;t't'_z} = \sum_{t''t''_z}\sqrt{\frac{[2t+1]}{[2t''+1]}}\,(jj_z\,tt_z|t''t''_z)_q\, U(jj't''t';j+j't)_q\,\mathbf{IR}^{j+j'(q)}_{t''t''_z;t't'_z}. \tag{3.13}$$

Here $U(\ldots;\ldots)_q$ is recoupling coefficient which can be expressed via the stretched q-$6j$-symbols of $U_q(su_T(2))$ [5]:

$$U(jj't''t';j+j't)_q = (-1)^{j+j'+t'+t''}\sqrt{[2j+2j'+1][2t+1]}\begin{Bmatrix} j & j' & j+j' \\ t' & t'' & t\end{Bmatrix}_q. \tag{3.14}$$

Using (3.9) and (3.10) we can present the basis vectors (2.13) in the form of

$$|(\lambda\mu)jtt_z\rangle = \mathcal{F}_-^{(\lambda\mu)}(jtt_z)|(\lambda\mu)h\rangle = \mathcal{N}^{(\lambda\mu)}_{jt}\,\mathbf{IR}^{j(q)}_{tt_z;\frac{1}{2}\mu\frac{1}{2}\mu}|(\lambda\mu)h\rangle\ . \tag{3.15}$$

The normalizing factor $\mathcal{N}^{(\lambda\mu)}_{jt}$ is given by

$$\begin{aligned}\mathcal{N}^{(\lambda\mu)}_{jt} &= (-1)^{2j} q^{(j+\frac{1}{2}\mu-t)(j-\frac{1}{2}\mu+t)+j\lambda+\frac{1}{2}\mu(j+\frac{1}{2}\mu-t)-2j^2+j+t-\frac{1}{2}\mu} \\ &\quad\times\sqrt{\frac{[j-\frac{1}{2}\mu+t]![j+\frac{1}{2}\mu-t]!}{[2j]![\mu+1]}}\,(j\,\tfrac{1}{2}\mu-t\;tt|\tfrac{1}{2}\mu\tfrac{1}{2}\mu)_q\; N^{(\lambda\mu)}_{jt}\ .\end{aligned} \tag{3.16}$$

With the help of (3.15) we easy find the action of the ITO (3.10) on the Gelfand-Tsetlin basis:

$$R^{j'(q)}_{j'_z}|(\lambda\mu)jtt_z\rangle = \sum_{t''t''_z}(j'j'_z\,tt_z|t''t''_z)_q\,\langle(\lambda\mu)j''t''\|R^{j'(q)}\|(\lambda\mu)jt\rangle_q\,|(\lambda\mu)j''t''t''_z\rangle\ , \tag{3.17}$$

where

$$\langle(\lambda\mu)j''t''\|R^{j'(q)}\|(\lambda\mu)jt\rangle_q = \delta_{j'',j+j'}\sqrt{\frac{[2t+1]}{[2t''+1]}}\,\frac{\mathcal{N}^{(\lambda\mu)}_{jt}}{\mathcal{N}^{(\lambda\mu)}_{j''t''}}\,U(j'j\,t''\,\tfrac{1}{2}\mu;j'+jt)_q\ . \tag{3.18}$$

4 Tensor form of the projection operator

It is obvious that the extremal projector (2.5) can be presented in the form

$$p(U_q(su(3))=p(U_q(su_T(2))(p_{12}p_{13})p(U_q(su_T(2)). \tag{4.1}$$

Now we present the middle part of (4.1) in the terms of the $U_q(su_T(2))$ tensor operators (3.10) and (3.11). To this end, we substitute the explicit expression (2.6) for the factors p_{12} and p_{13}, and combine monomials $e_{21}^n e_{31}^m$ and $e_{12}^n e_{13}^m$. After some summation manipulations we obtain the following expression for the extremal projection operator $p:=p(U_q(su(3)))$ in terms of the tensor operators (3.10) and (3.11):

$$p=p(U_q(su_T(2))\Big(\sum_{jj_z}A_{jj_z}\tilde{R}_{j_z}^{j(q)}R_{j_z}^{\prime j(q)}\Big)p(U_q(su_T(2)). \tag{4.2}$$

Here

$$A_{jj_z}=\frac{(-1)^{3j}[\varphi_{12}][\varphi_{12}+j+j_z-1]![\varphi_{13}]!}{[2j]![\varphi_{12}+2j]![\varphi_{13}+j+j_z]!}\,q^{4j^2+j+2jh_{\alpha_1}+2(j+j_z)T_0}, \tag{4.3}$$

where $\varphi_{1i+1}:=e_{11}-e_{i+1i+1}+i$ $(i=1,2)$. Below we assume that the $U_q(su(3))$ extremal projection operator p acts in a weight space with the weight $(\lambda\mu)$ and in this case the symbol p is supplied with the index $(\lambda\mu)$, $p^{(\lambda\mu)}$, and all the Cartan elements h_{α_i} on the right side of (4.2) are replaced by the corresponding weight components λ and μ. Now we multiple the projector $p^{(\lambda\mu)}$ from the left side by the lowering operator $\mathcal{F}_-^{(\lambda\mu)}(jtt_z)$ and from the right side by the rising operator $(\mathcal{F}_-^{(\lambda\mu)}(jtt_z))^*$, and by applying a relation of type (3.13) we finally find the tensor form of the general $U_q(su(3))$ projection operator:

$$P_{jtt_z;j't't'_z}^{(\lambda\mu)}=\sum_{j''t''}B_{j''t''}^{(\lambda\mu)}\,\mathbb{R}_{tt_z,t''t''}^{j+j''(q)}\,\mathbb{R}_{t''t'',t't'_z}^{\prime j''+j'(q)}\ , \tag{4.4}$$

were the coefficients $B_{j''t''}^{(\lambda\mu)}$ are given by

$$\begin{aligned}B_{j''t''}^{(\lambda\mu)}&=\frac{(-1)^{2j+j'+j''-t'+t''}q^{\phi}[\lambda+1][\mu+1][\lambda+\mu+2]}{[\lambda+\frac{1}{2}\mu+j''+t''+2]![\lambda+\frac{1}{2}\mu+j''-t''+1]![2j'']!}\begin{Bmatrix}j & j'' & j+j''\\ t'' & t & \frac{1}{2}\mu\end{Bmatrix}_q\begin{Bmatrix}j' & j'' & j'+j''\\ t'' & t' & \frac{1}{2}\mu\end{Bmatrix}_q\\&\times\Big(\frac{[\lambda+\frac{1}{2}\mu-j+t+1]![\lambda+\frac{1}{2}\mu-j-t]![\lambda+\frac{1}{2}\mu-j'+t'+1]![\lambda+\frac{1}{2}\mu-j'-t']![2j+2j''+1][2j'+2j''+1]}{[2j]![2j']![2t+1][2t'+1]}\Big)^{\frac{1}{2}},\end{aligned} \tag{4.5}$$

$$\phi=\varphi(\lambda,\mu,j,t)+\varphi(\lambda,\mu,j',t')-2\varphi(\lambda,\mu,j'',t'')+j''(4\lambda+2\mu-1)+4t''-2\mu-3j'. \tag{4.6}$$

Here and elsewhere we use the notation $\varphi(\lambda,\mu,j,t):=\frac{1}{2}(\frac{1}{2}\mu+j-t)(\frac{1}{2}\mu+j+t-3)+j(\lambda-2j+1)$.

5 General form of Clebsch-Gordan coefficients

For convenience we introduce the short notations: $\Lambda:=(\lambda\mu)$ and $\gamma:=jtt_z$ and therefore the basis vector $|(\lambda\mu)jtt_z\rangle$ will be is denoted by $|\Lambda\gamma\rangle$. Let $\{|\Lambda_i\gamma_i\rangle\}$ be bases of two IRs Λ_i $(i=$

1, 2). Then $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$ be a basis in the representation $\Lambda_1\otimes\Lambda_2$ of $U_q(su(3))\otimes U_q(su(3))$. In this representation there is an another coupled basis $|\Lambda_1\Lambda_2 : s\Lambda_3\gamma_3\rangle_q$ with respect to $\Delta_q(U_q(su(3)))$ where the index s classifies multiple representations Λ_3.We can expand the coupled basis in terms of the tensor uncoupled basis $\{|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\}$:

$$|\Lambda_1\Lambda_2\colon s\Lambda_3\gamma_3\rangle_q=\sum_{\gamma_1,\gamma_2}(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|s\Lambda_3\gamma_3)_q\,|\Lambda_1\gamma_1\rangle|\Lambda_2\gamma_2\rangle\ , \tag{5.1}$$

where the matrix element $(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|s\Lambda_3\gamma_3)_q$ is the Clebsch-Gordan coefficient of $U_q(su(3))$. In just the same way as for the non-quantized Lie algebra $su(3)$ (see [3]) we can show that any CGC of $U_q(su(3))$ can be represented in terms of the linear combination of the matrix elements of the projection operator (4.4)

$$(\Lambda_1\gamma_1\,\Lambda_2\gamma_2|s\Lambda_3\gamma_3)_q=\sum_{\gamma'_2}C(\gamma'_2)\,\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,h})|\Lambda_1h\rangle|\Lambda_2\gamma'_2\rangle\ . \tag{5.2}$$

Classification of multiple representations Λ_3 in the representation $\Lambda_1\otimes\Lambda_2$ is special problem and we shall not touch it here. For the non-deformed algebra $su(3)$ this problem was considered in details in [3]. Concerning of the matrix elements in the right-side of (5.2) we give here an explicit expression for the more general matrix element:

$$\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,\gamma'_3})|\Lambda_1\gamma'_1\rangle|\Lambda_2\gamma'_2\rangle\ . \tag{5.3}$$

Using (4.4) and the Wigner-Racah calculus for the subalgebra $U_q(su(2))$ [5] (analogously to the non-quantized Lie algebra $su(3)$ [3]) it is not hard to obtain the following result:

$$\begin{aligned}\langle\Lambda_1\gamma_1|\langle\Lambda_2\gamma_2|\Delta_q(P^{\Lambda_3}_{\gamma_3,\gamma'_3})|\Lambda_1\gamma'_1\rangle|\Lambda_2\gamma'_2\rangle &= (t_1t_{1z}\,t_2t_{2z}|t_3t_{3z})_q\,(t_1t'_{1z}\,t_2t'_{2z}|t'_3t'_{3z})_q\\ &\times[\lambda_3+1][\mu_3+1][\lambda_3+\mu_3+2]\;A\sum_{j''_1j''_2t''_1t''_2t''_3}C_{j''_1j''_2t''_1t''_2t''_3}\\ &\times\left\{\begin{array}{ccc} j_1-j''_1 & j_2-j''_2 & j_1+j_2-j''_1-j''_2\\ t''_1 & t''_2 & t''_3\\ t_1 & t_2 & t_3\end{array}\right\}_q\left\{\begin{array}{ccc} j'_1-j''_1 & j'_2-j''_2 & j'_1+j'_2-j''_1-j''_2\\ t''_1 & t''_2 & t''_3\\ t'_1 & t'_2 & t'_3\end{array}\right\}_q .\end{aligned} \tag{5.4}$$

Here

$$\begin{aligned}A &= \Bigl(\frac{[2t_1+1][2t_2+1][2j_1+1]![2j_2+1]![\lambda_3+\frac{1}{2}\mu_3-j_3+t_3+1]![\lambda_3+\frac{1}{2}\mu_3-j_3-t_3]!}{[\lambda_1+\frac{1}{2}\mu_1-j_1+t_1+1]![\lambda_1+\frac{1}{2}\mu_1-j_1-t_1]![\lambda_2+\frac{1}{2}\mu_2-j_2+t_2+1]![\lambda_2+\frac{1}{2}\mu_2-j_2-t_2]![2j_3]!}\\ &\times\frac{[2t'_1+1][2t'_2+1][2j'_1+1]![2j'_2+1]![\lambda_3+\frac{1}{2}\mu_3-j'_3+t'_3+1]![\lambda_3+\frac{1}{2}\mu_3-j'_3-t'_3]!}{[\lambda_1+\frac{1}{2}\mu_1-j'_1+t'_1+1]![\lambda_1+\frac{1}{2}\mu_1-j'_1-t'_1]![\lambda_2+\frac{1}{2}\mu_2-j'_2+t'_2+1]![\lambda_2+\frac{1}{2}\mu_2-j'_2-t'_2]![2j'_3]!}\Bigr)^{\frac{1}{2}},\end{aligned} \tag{5.5}$$

$$
\begin{aligned}
C_{j_1''j_2''t_1''t_2''t_3''} &= \frac{(-1)^{2(j_1+j_2+j_3'-j_1''-j_2'')}\, q^{\psi}[2(j_1+j_2-j_1''-j_2'')+1]![2(j_1'+j_2'-j_1''-j_2'')+1]!}{[2j_1'']![2j_2'']![2j_1-2j_1'']![2j_2-2j_2'']![2j_1'-2j_2'']![2j_2'-2j_2'']![2(j_1+j_2-j_3-j_1''-j_2'')]!} \\
&\times \frac{[\lambda_1+\frac{1}{2}\mu_1-j_1''+t_1''+1]![\lambda_1+\frac{1}{2}\mu_1-j_1''-t_1'']![\lambda_2+\frac{1}{2}\mu_2-j_2''+t_2''+1]![\lambda_2+\frac{1}{2}\mu_2-j_2''-t_2'']![2t_1''+1][2t_2''+1][2t_3''+1]}{[\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_1''-j_2''+t_3''+2]![\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_1''-j_2''-t_3''+1]!} \\
&\times \left\{\begin{matrix} j_1-j_1'' & j_1'' & j_1 \\ \frac{1}{2}\mu_1 & t_1 & t_1'' \end{matrix}\right\}_q \left\{\begin{matrix} j_2-j_2'' & j_2'' & j_2 \\ \frac{1}{2}\mu_2 & t_2 & t_2'' \end{matrix}\right\}_q \left\{\begin{matrix} j_3 & j_1+j_2-j_3-j_1''-j_2'' & j_1+j_2-j_1''-j_2'' \\ t_3'' & t_3 & \frac{1}{2}\mu_3 \end{matrix}\right\}_q \\
&\times \left\{\begin{matrix} j_1'-j_1'' & j_1'' & j_1' \\ \frac{1}{2}\mu_1 & t_1' & t_1'' \end{matrix}\right\}_q \left\{\begin{matrix} j_2'-j_2'' & j_2'' & j_2' \\ \frac{1}{2}\mu_2 & t_2' & t_2'' \end{matrix}\right\}_q \left\{\begin{matrix} j_3' & j_1'+j_2'-j_3'-j_1''-j_2'' & j_1'+j_2'-j_1''-j_2'' \\ t_3'' & t_3' & \frac{1}{2}\mu_3 \end{matrix}\right\}_q ,
\end{aligned}
\tag{5.6}
$$

where $\psi = \sum_{i=1}^{2}\Big(2\varphi(\lambda_i,\mu_i,j_i'',t_i'') - \varphi(\lambda_i,\mu_i,j_i,t_i) - \varphi(\lambda_i,\mu_i,j_i',t_i') - t_i(t_i+1) - t_i'(t_i'+1)\Big)$ $- 2\varphi(\lambda_3,\mu_3,j_3'',t_3'') + \varphi(\lambda_3,\mu_3,j_3,t_3) + \varphi(\lambda_3,\mu_3,j_3',t_3') + j_3''(4\lambda_3+2\mu_3+2) - 2t_3''(t_3''-1) - 2\mu_3$ $- (j_2+j_2'-2j_2'')(2\lambda_1+\mu_1-6j_1'') + 4(j_1-j_1'')(j_2-j_2'') + 4(j_1'-j_1'')(j_2'-j_2'') - (j_3+j_3'')(j_3+j_3''+1)$ $- (j_3'+j_3'')(j_3'+j_3''+1)$, $j_3'' := j_1+j_2-j_3-j_1''-j_2'' = j_1'+j_2'-j_3'-j_1''-j_2''$.

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